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Probability distribution of persistent spins in an Ising chain

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Abstract

We study the probability distribution $Q(n, t)$ of $n(t)$, the fraction of spins unflipped up to time t, in an Ising chain with ferromagnetic interactions. The distribution shows a peak at $n = n_{\rm max}$ and in general is non-Gaussian and asymmetric in nature. However, for $n > n_{\rm max}$ it shows a Gaussian decay. Data collapse can be obtained when $Q(n, t)/L^{\alpha}$ versus $(n - n_{\rm max})L^{\beta}$ is plotted with $\alpha \sim 0.45$ and $\beta \sim 0.6$. Interestingly, $n_{\rm max}(t)$ shows different behaviour compared to $\langle n(t)\rangle = P(t)$, the persistence probability which follows the well-known behaviour $P(t) \sim t^{-\theta}$. A quantitative estimate of the asymmetry and non-Gaussian nature of $Q(n, t)$ is made by calculating its skewness and kurtosis.

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In recent years a lot of work has been devoted to studying 'persistence' in dynamical systems [1]. Persistence is the phenomenon defined as the probability that a fluctuating non-equilibrium field has not changed its sign up to time t. This phenomenon has been observed in magnetic systems [2–6], simple diffusion [7], coarsening dynamics [8], various models undergoing phase separation process [9], fluctuating interfaces [10] etc.

In Ising system, persistence is simply the probability that a spin has not changed its sign up to time t after the system is quenched to a low temperature from an initial high temperature. The fraction of the persistent spins $P(t)$ here is given by

$$P(t) \sim t^{-\theta} \tag{1}$$

where θ is a new exponent not related to any previously known static or dynamic exponent. In one dimension, at $T = 0$, θ is exactly known, $\theta = 0.375$ [2]. In higher dimensions, the persistence exponent has been obtained approximately using both analytical and numerical methods.

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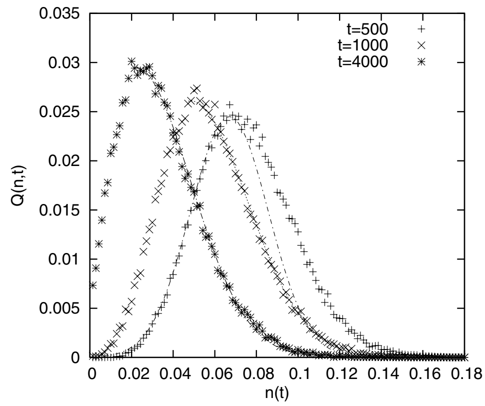

Figure 1. Probability distribution $Q(n,t)$ as a function of n at $t = 500$, 1000 and 4000 for $L = 700$. The continuous lines are Gaussian fits of the form $\exp(-(x - x_0)^2/\alpha)$. For the fits shown for $t = 1000$ and 4000 on the right side of the peak, $\alpha = \alpha_r = 1.5 \times 10^{-3}$ and 1.4×10^{-3}, respectively. The left side of the peak can be fitted to the same form with a different value of α: $\alpha = \alpha_l = 7.0 \times 10^{-4}$ as shown in the curve for $t = 500$.

In the numerical studies, one needs to generate different random initial configurations to obtain the persistence probability $P(t)$ which is an averaged out quantity. We define $n(t)$ to be the fraction of persistent spins up to time t which has different values for different realizations of randomness such that $\langle n(t)\rangle = P(t)$, where $\langle\rangle$ denotes average over realizations. Thus $n(t)$ can be defined as a stochastic variable described by a probability distribution function. We have precisely studied the probability distribution $Q(n,t)$ of $n(t)$ and obtained a number of interesting features of the distribution for the one-dimensional Ising model. The probability distribution of random variables makes for interesting studies in various systems, e.g., for the mass of spanning clusters in percolation [11], random Ising and bond diluted Ashkin–Teller model [12], conductance of classical dilute resistor network [13], directed polymers and growth models [14], degree distribution in networks [15] etc. In several of these systems, the distribution is non-Gaussian and shows many interesting features. Certain properties such as self-averaging, multifractality etc can be studied directly from the distribution function [16]. Also measurements such as skewness and kurtosis [17] from the higher moments to estimate quantitatively the asymmetry and departure from Gaussian behaviour of the distribution are possible.

We have considered a chain of Ising spins with nearest-neighbour ferromagnetic interaction and simulated it using periodic boundary conditions. The interaction is represented by the Hamiltonian

$$H = -J \sum_i s_i s_{i+1}. \tag{2}$$

The initial configuration is random and single spin flip (deterministic) Glauber dynamics has been used for subsequent updating.

Since a large number of configurations are required to obtain accurate data, the system sizes are restricted to $L \leqslant 1000$. Primarily, we are interested in the form of the distribution $Q(n,t)$ which is plotted at different times against $n(t)$ in figure 1. We have obtained $Q(n,t)$ for several values of L also.

At all times, the distribution $Q(n,t)$ shows a peak at $n(t) = n_{\max}(t)$. The value of $n_{\max}(t)$ has negligible dependence on L. At early times, there is a Gaussian decay of $Q(n,t)$ on both

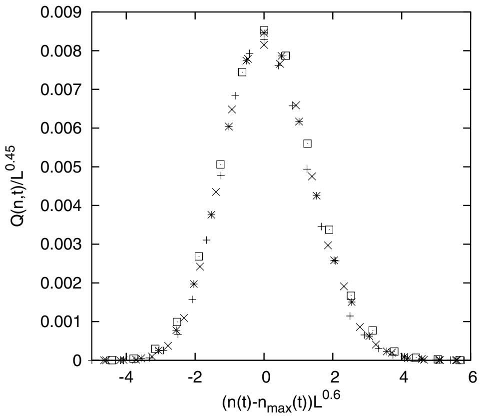

Figure 2. Scaled probability distribution $Q(n,t)/L^{0.45}$ as a function of $(n(t) - n_{\max})L^{0.6}$ at $t = 200$ for $L = 500$, 600, 700 and 1000 shows a collapse.

sides of the peak. Interestingly, the Gaussian behaviour $\exp(-(x - x_0)^2/\alpha)$ is followed with different α values on the two sides of $n_{\max}$, α_l on the left and α_r on the right. For very large t, α_r is the only measure possible as it is difficult to fit the function to a Gaussian on the left side of the peak. Usually the decay behaviour of distributions for rare events is of interest and we find the decay is Gaussian at all times for $n > n_{\max}$. We observe that α_r shows a weak dependence on t which becomes negligible for larger system sizes. It is also a function of L, $\alpha_r \sim L^{-1.2}$. α_l, which can be calculated accurately for initial times, follows a similar scaling. In fact, the scaled distribution $Q(n,t)/L^{\alpha}$ plotted against $(n(t) - n_{\max}(t))L^{\beta}$ with $\alpha \sim 0.45$ and $\beta \sim 0.6$ shows a nice data collapse (figure 2). Even at long times, a fairly good data collapse can be obtained with these values of α and β.

The distribution has natural cut-offs at $n = 0$ and $n = 1$. Therefore as time evolves, the distribution becomes more and more asymmetric as the fraction of persistent spins decreases with time. This asymmetry is more apparent when the probability that there is no persistent spin, $Q(0,t)$, begins to assume finite values.

In fact the behaviour of $Q(0,t)$ is quite interesting itself. In figure 3, $Q(0,t)$ has been plotted against t/L^2 and the data for different L values seem to fall on the same curve indicating that $Q(0,t)$ is a function of t/L^2 with the behaviour

$$\begin{aligned} Q(0,t) &= 0 \qquad \text{for} \quad t/L^2 < a_0 \\ &\neq 0 \qquad \text{for} \quad t/L^2 > a_0 \end{aligned}$$

where $a_0 \sim 0.001$. While t/L^2 appearing as a scaling argument is expected, what is notable is the small value of the threshold a_0.

The comparison of the behaviour of the most probable value $n_{\max}(t)$ and the average value $P(t)$ shows consistent features. In figure 4, we have plotted both $P(t)$ and $n_{\max}(t)$ against t. While $P(t)$ shows the expected power law decay with the known exponent $\theta = 0.375$, $n_{\max}(t)$ shows a different behaviour. $n_{\max}(t)$ falls off faster than $P(t)$ and it is not possible to fit a power law to it.

One can try a power law fit only for very early times with a value of θ close to 0.375. For $P(t)$ it is known that the behaviour $P(t) \sim t^{-\theta}$ is valid for $t < \tau$, where $\tau \sim L^2$. In the case that the power law behaviour of $n_{\max}$ is valid for a finite time in the same sense, it appears that the deviation from the power law takes place at a much earlier time. We find that the so-called deviation occurs at a value of $t/L^2 \sim a_0$ indicating that the most probable value fails

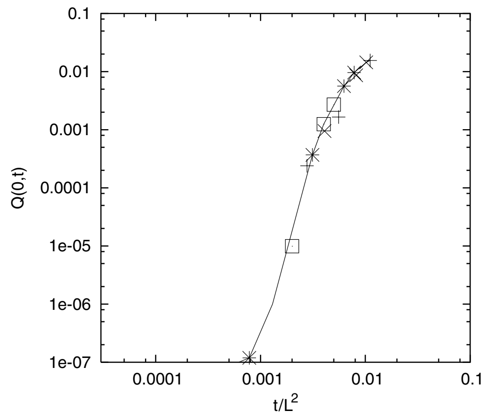

Figure 3. $Q(0,t)$ as a function of t/L^2 for four different L values. The solid line is a guide to the eye.

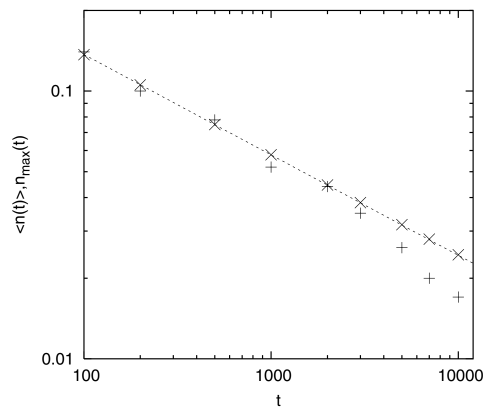

Figure 4. $\langle n(t)\rangle = P(t)$ ($\times$) and $n_{\max}(t)$ (+) are plotted against time t for $L = 1000$. $\langle n(t)\rangle(t) \sim t^{-0.375} n_{\max}(t)$ shows deviation from this behaviour at finite times (the dashed line with slope -0.375 is fitted to $\langle n(t)\rangle$).

to show persistence behaviour when $Q(0,t)$ becomes non-zero. Also $P(t)$ for $t \to \infty$ goes to a constant value ($\sim L^{-2\theta}$) while $n_{\max}$ goes to zero for $t \to \infty$.

One can easily calculate the higher moments from the probability distribution. We have studied the self-averaging property and also tried to estimate the asymmetry and the non-Gaussian behaviour by calculating the skewness and kurtosis of the distribution.

A system is said to exhibit self-averaging if $R_x(L) = (\Delta x)^2/\langle x\rangle^2 \to 0$ as $L \to \infty$ for any physical quantity x [12, 16] (Δx is the variance $= \langle x^2\rangle - \langle x\rangle^2$). Here we have calculated $R_n(L) = (\Delta n)^2/\langle n(t)\rangle^2$ to check whether self-averaging is present. Our results show that $R_n(L,t) \sim L^{-\theta_l}$ where $\theta_l \sim 0.51$ indicating strong self-averaging. In fact $R_n(L,t)$ also shows a power law increase with time such that $R_n(L,t)/t^{\theta_t}$ for different values of L shows a collapse (figure 5). Apparently, the variance is a weak function of time: $(\Delta n)^2 \sim t^{-0.06}$.

To measure the skewness, we calculate $s(t,L) = M_3/(M_2)^{3/2}$ where the mth centred moment is $M_m = \langle (x - \langle x\rangle)^m\rangle$. s measures the asymmetry of the distribution and is zero for a symmetric distribution. Here it not only shows an increase with time as expected (e.g., for $L = 1000$ $s \sim 0.18$, 0.36 and 0.57 for $t = 200$, 500 and 2000, respectively), but shows a

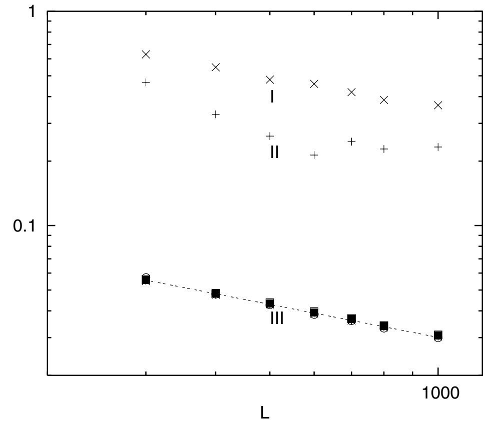

Figure 5. The skewness (curve marked I), kurtosis (II) and scaled R_n (III) at $t = 500$ as a function of the system size L have been shown. R_n/t^{θ_t} (data for four different values of time have been plotted) is seen to vary as $L^{-\theta_l}$ with $\theta_l \sim 0.51$ and $\theta_t \sim 0.345$.

decay with L as well. However, the dependence on L weakens at longer times. We expect s to remain finite at $L \to \infty$, which is not so apparent from the data presumably because of the small system sizes considered. That there is no universal distribution with respect to time as in [10] is reflected by the fact that s is time dependent.

Kurtosis is a measure of the peakedness of the distribution. It is studied by calculating $k(t, L) = M_4/(M_2)^2 - 3$. For a Gaussian distribution, $k(t, L) = 0$ indicating that the peak is at the mean value. A negative value of k would imply that the distribution is flat. k shows a saturation with L for all times indicating the non-Gaussian behaviour of the distribution. It has a positive value $\sim$0.2 to show that the distribution is peaked close to the mean. This value is also independent of time implying that non-Gaussian behaviour remains constant quantitatively with time. In figure 5, typical variations of s and k with L have been shown.

In summary, we have obtained the distribution function for the fraction of persistent spins $n(t)$ for a one-dimensional ferromagnetic Ising system. The form of the distribution is non-Gaussian in general. The form also changes with time, becoming more and more asymmetric at longer times. The most probable value $n_{\max}(t)$ shows deviation from the average $\langle n(t)\rangle = P(t) \sim t^{-\theta}$ at times $t/L^2 > a_0$ where $a_0 \sim 10^{-3}$. Here we also find that $Q(0, t)$ begins to take non-zero values. The system also shows strong self-averaging.

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